JOURNAL OF APPROXIMATION THEORY 82, 240–256 (1995)

Coordinate Order of Approximation by Functional-Based Approximation Operators

HERMANN G. BURCHARD

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078

AND

JUNJIANG LEI

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901

Communicated by Will Light

Received July 23, 1993; accepted in revised form August 2, 1994

We study coordinatewise L_p approximation by a fairly general class of linear operators that includes quasi-intrpolants and, like these, is based on a globally supported basis function and a globally supported linear functional of general form with certain mild decay conditions imposed on the basis function and the functional involved. In this quite general setting we show that the approximation power provided by a quasi-interpolant and other functional-based operators is equivalent to the polynomial reproducing property possessed by it. © 1995 Academic Press, Inc.

1. INTRODUCTION

The study of approximation from a principal shift invariant (PSI) space [BDR] has recently attracted much attention, because it applications are found in many fields such as radial basis functions and wavelets. The scheme of quasi-interpolation plays an important role in this study, where we treat a fairly general scheme based on a general functional. In the present paper we are interested in the coordinate power provided by a PSI space through a functional-based operator of the form

$$Q_{\phi,\lambda} f := \sum_{j \in \mathbb{Z}^d} \lambda f(\cdot + j)\, \phi(\cdot - j),$$

where λ is a suitable linear functional. Obviously, this includes the cases usually designated as quasi-interpolants, which have been given a

0021-9045/95 $12.00

systematic treatment in an elegant survey paper [B_1] by de Boor, where ϕ is assumed to be a compactly supported function and λ a compactly supported distribution. Recently several authors have also considered the approximation power provided by $Q_{\phi,\lambda}$ with ϕ being globally supported (e.g., [LC], [JL], and [HL]). They assume that the functional λ is a point-evaluation functional, or a modified version of point-evaluation. However, an observation made in [L_2] shows that sometimes one may wish to have a quasi-interpolant in which λ and ϕ are both globally supported and λ is a linear functional other than any kind of point-evaluations. From the experience with compactly supported ϕ and λ, it is natural to expect and believe that the equivalence between the polynomial reproducing property of $Q_{\phi,\lambda}$ and the approximation power provided by $Q_{\phi,\lambda}$ (cf. [B_2]) be valid in a quite general setting. One of the goals of this paper is to confirm this point by studying the functional-based approximation operators $Q_{\phi,\lambda}$ in which λ may be any linear functional on $L_p(\mathbb{R}^d)$ ($1 \leqslant p < \infty$) or on C_0, the space of continuous functions vanishing at infinity, as long as λ and ϕ satisfy certain mild decay conditions. Besides, in this way an opportunity is provided for us to unify and improve results and methods of [JL], [L_1] and [L_2] in estimating the L_p approximation power of quasi-interpolants.

The study of coordinate order of approximation refers to coordinate-wise scaling of approximations

$$Qf = \sum_{j \in \mathbb{Z}^d} \lambda f(h(\cdot + j))\, \phi(\cdot/h - j),$$

with $h \in \mathbb{R}^d_+$ a vector of positive step-sizes, letting $\|h\| \to 0$. We illustrate the differences between coordinate order and the more familiar total order by means of an example, which exhibits interesting new phenomena which do not show up when attention is restricted to total order [BrCW]. Special techniques are required in the proofs, e.g., a new Taylor's formula with integral remainder. Approximation in a coordinatewise setting was also studied by several other authors, e.g., [DDS] and [CL].

We would also like to mention that some methods of approximation other than functional-based operators have recently been developed, cf. [BRD], [BR] and [B_2].

Next, let us fix some notation to be used in this paper. We denote by $\|x\|$ the Euclidean norm of x, by $x \cdot y$ the usual dot product of $x, y \in \mathbb{R}^d$, and monomials by $x^\alpha = x_1^{\alpha_1} \cdot \cdots \cdot x_d^{\alpha_d}$. We have already referred to coordinate-wise scaling by a vector $h \in \mathbb{R}^d_+$ of positive step-sizes, this involves operations $xh := (x_1 h_1, ..., x_d h_d)$, and $x/h := (x_1/h_1, ..., x_d/h_d)$. Here, $\mathbb{R}^d_+$ denotes the subset of $\mathbb{R}^d$ of vectors having non-negative components, also $\mathbb{Z}^d_+ := \mathbb{Z}^d \cap \mathbb{R}^d_+$, and by $B_r(x)$ denote the closed ball of radius $r > 0$ centered at x. For the study of the coordinate order of approximation let K be a lower set, i.e., a finite subset of $\mathbb{Z}^d_+$ with the property that $\beta \in K$ implies

$\alpha \in K$ for all $\alpha \in \mathbb{Z}_+^d$ with $\alpha \leqslant \beta$. Define Π_K to be te span of the monomials x^α, $\alpha \in K$. Let μ be a regular Borel measure on $\mathbb{R}^d$, $|\mu|$ its total variation measure and $L_1(|\mu|)$ the space of $|\mu|$-integrable functions. In order to get operators that are able to act on Π_K, we shall always assume that μ decays in such a way that $\Pi_K \subset L_1(|\mu|)$. In this case the Fourier transform of μ

$$\hat{\mu}(x) := \int_{\mathbb{R}^d} e^{ix \cdot y} \, d\mu(y), \qquad x \in \mathbb{R}^d$$

is a continuous function and has continuous derivatives $D^\alpha \hat{\mu}$ for $\alpha \in K$. This includes the case of the Fourier transform of a function in $L_1(\mathbb{R}^d)$, subject to the above decay condition. If a function $g \in L_p(\mathbb{R}^d)$ $(1 \leqslant p < \infty)$ decays in a way such that $\sum_{j \in \mathbb{Z}^d} |g(\cdot + j)|$ is in $L_p([0, 1)^d)$ $(1 \leqslant p < \infty)$, then g will be said to be in $\mathscr{L}_p$ (cf. [JM]).

2. The Main Results

Our first result is concerned with the equivalence between the Strang–Fix conditions and a polynomial reproducing property under the fairly general conditions which we now state precisely. A function ϕ in $L_1(\mathbb{R}^d)$ with the decay property that $\phi \Pi_K \subset L_1(\mathbb{R}^d)$ is said to satisfy the Strang–Fix conditions on K if

$$\hat{\phi}(0) \neq 0$$

and

$$D^\alpha \hat{\phi}(2\pi j) = 0 \qquad \text{for} \quad \alpha \in K \quad \text{and} \quad j \in \mathbb{Z}^d \backslash \{0\}.$$

THEOREM 2.1. *Let ϕ be a function such that $\phi \Pi_K \subset L_1(\mathbb{R}^d)$, and μ be a regular Borel measure on $\mathbb{R}^d$ such that $\Pi_K \subset L_1(|\mu|)$. The following are equivalent*:

(a) $\sum_{j \in \mathbb{Z}^d} \mu(f(\cdot + j)) \, \phi(\cdot - j) = f$ *for all* $f \in \Pi_K$, *with* $\mu(g) := \int_{\mathbb{R}^d} g \, d\mu$;

(b) ϕ *satisfies the Strang–Fix conditions on K and the conditions that*

$$\hat{\phi}(0) \, \hat{\mu}(0) = 1 \qquad \text{and} \qquad D^\alpha(\hat{\phi}\hat{\mu})(0) = 0 \qquad \text{for} \quad 0 \neq \alpha \in K.$$

In case that ϕ is compactly supported and μ is an atomic measure, this result was apparently given in [CL] and essentially in [CJW]. For compactly supported ϕ and μ, it can also be obtained from [Theorem 5.10, B_1]. Our proof of Theorem 2.1 will be given in the next section.

Let K^+ be the extension of a lower set K which by definition is a union of K and the set $\{\alpha+e^i : \alpha \in K, i=1, ..., d\}$ with $\{e^1, ..., e^d\}$ being the standard basis of $\mathbb{R}^d$. Define the "boundary" ∂K as follows:

$$\partial K := K^+ \backslash K.$$

We also recall that for a bounded linear functional λ belonging to the dual space X' of $X = C_0(\mathbb{R}^d)$ for $p=\infty$, or of $X = L_p(\mathbb{R}^d)$ for $1 \leqslant p < \infty$, there exists a unique regular Borel measure μ_λ for which the representation $\lambda(g) = \int g \, d\mu_\lambda$ holds. Of course, for $X = L_p(\mathbb{R}^d)$ $(1 \leqslant p < \infty)$ we can write $d\mu_\lambda = f_\lambda \, d\mu$, with $f_\lambda \in X' = L_q(\mathbb{R}^d)$, $1/p + 1/q = 1$.

THEOREM 2.2. *Let ϕ be a function such that $\phi \Pi_{K^+} \subset \mathscr{L}_p$ $(1 \leqslant p \leqslant \infty)$. Let λ belong to the dual space of $L_p(\mathbb{R}^d)$ for $1 \leqslant p < \infty$ (or of $C_0(\mathbb{R}^d)$ for $p = \infty$ respectively), such that $\Pi_{K^+} \subset L_1(|\mu_\lambda|)$. If (a) or (b) (hence both) of Theorem 2.1 hold for $\mu = \mu_\lambda$, then*

$$\left\| f - \sum_{j \in \mathbb{Z}^d} \lambda f(h(\cdot + j)) \, \phi(\cdot/h - j) \right\|_{L_p(\mathbb{R}^d)} = \mathcal{O}(\max\{h^\beta : \beta \in \partial K\})$$

as $\|h\| \to 0$, for every f in the Sobolev space $W_{m,p}(\mathbb{R}^d)$ (or in $W_{m,\infty} \cap C_0$ for $p = \infty$), where m is the maximum of $|\beta|$ for $\beta \in \partial K$.

The proof is given in Section 4. Recall that the notations $f(h(\cdot + j))$ and $\phi(\cdot/h)$ refer to coordinate-wise scaling operations by the vector $h \in \mathbb{R}^d_+$ of positive stepsizes. The result of Theorem 2.2 for total order of approximation by means of translates of a globally supported function ϕ under somewhat more restrictive conditions on the functional λ as well as the function ϕ had been obtained in [LC], [JL] and [L_1]. For a coordinate version cf. [BrCW] (also [CJW] and [CL]).

The conditions of the theorem are satisfied by radial basis functions, in particular multiquadrics, cf. [Buh]. Although their rotational symmetry makes the index set uninteresting from the coordinate-order point of view, recently nonisotropic radial basis functions have been studied [Light].

To state our inverse result we need the lower set K to be concave, which means that K has empty intersection with the convex hull of $\mathbb{Z}^d_+ \backslash K$ (cf. [BrCW], for an interesting characterization of this condition).

THEOREM 2.3. *Adopt the hypothesis of Theorem 2.2, and assume in addition that the lower set K is concave. If there exists any $r > 0$ and $z \in \mathbb{R}^d$ such that*

$$\left\| f - \sum_{j \in \mathbb{Z}^d} \lambda f(h(\cdot + j)) \, \phi(\cdot/h - j) \right\|_{L_p(B_r(z))} = \mathcal{O}(\max\{h^\beta : \beta \in \partial K\}),$$

as $\|h\| \to 0$, *for every* $f \in W_{m,p}(\mathbb{R}^d)$, *with m as in Theorem* 2.2, *then* (*a*) *and* (*b*) *of Theorem* 2.1 *hold, setting* $\mu = \mu_\lambda$.

We illustrate some of the differences between total and coordinate order of approximation by means of the following example.

EXAMPLE 2.4. Let r be a positive integer. Set

$$\psi = M_{(r+1,r,r+1)} + M_{(r,r+1,r+1)} - M_{(r+1,r+1,r)},$$

where $M_{(r,s,t)}$'s are the familiar box splines on a three direction mesh (cf. e.g., [Chui]). For simplicity of argument, we consider the approximation order provided by the PSI space generated by ψ in $L_2(\mathbb{R}^2)$ (cf. [BDR]). For this smooth (C^r), compactly supported function ψ,

$$K = \{\alpha\colon |\alpha| \leqslant 2r+1\} \setminus \{(r, r+1), (r+1, r)\}$$

is the maximal set on which the Strang–Fix conditions hold [BrCW]. It is easy to see that this lower set K is concave. Hence from Theorems 2.1 and 2.2 the coordinate order of approximation turns out to be (at least)

$$\mathcal{O}(\max\{h^\beta\colon \beta \in \partial K\}) = \mathcal{O}(h_1^r h_2^{r+1} + h_1^{r+1} h_2^r) + \mathcal{O}(\|h\|^{2r+2}).$$

The defect fom the desired but unachievable rate of $\mathcal{O}(\|h\|^{2r+2})$ is due to the two extra multi-integers $(r, r+1)$, $(r+1, r) \in \partial K$, which were missing from K. The total order $\mathcal{O}(h^{2r+1})$ is obtained by setting $h_2 = h_1$ (this total order may be computed directly from the Strang–Fix conditions for $\alpha \leqslant 2r+1$, $\alpha \in K$ by applying [Corollary 5.15, BDR]). However, the coordinate order can be made $\mathcal{O}(h_1^{2r+2})$ e.g., by setting $h_2 = h_1^{1+(1/r)}$. The defect becomes less and less significant for large r, a fact not discernible from the total order alone.

3. POLYNOMIAL REPRODUCING PROPERTY

In this section we prove Theorem 2.1. In the absence of compact support of ϕ and λ extra care is required in the verification of the equivalence between the Strang–Fix conditions and the polynomial reproducing property.

We denote by $g^{(\beta)}$ the derivative $D^\beta g$ for a $\beta \in \mathbb{Z}_+^d$ and g a sufficiently smooth function. For two functions f and g, we denote by $f *' g$ the semi-convolution $\sum_{j \in \mathbb{Z}^d} g(j) f(\cdot - j)$ and by $f * g$ the usual convolution.

LEMMA 3.1. *Let* ϕ *be a function such that* $\phi \Pi_K \subset L_1(\mathbb{R}^d)$, *and* f *be a polynomial in* Π_K. *The following are equivalent*:

(a) $\phi *' f$ *is a polynomial*;

(b) $f^{(\beta)}(-iD)\,\hat{\phi}(2\pi j)=0$ *for all* $j\in\mathbb{Z}^d\backslash\{0\}$ *and* $\beta\in\mathbb{Z}^d_+$;

(c) $\phi *' f=\phi * f$.

Proof. Before going to the circle of proof: (a)$\Rightarrow$(b), (b)$\Rightarrow$(c), (c)$\Rightarrow$(a), we give a general treatment of $\phi *' g$. Following (4.4) Proposition of [$\mathbf{B}_1$], we regard $\phi *' f$ as a tempered distribution and apply it to a test function $u\in\mathcal{S}$, the Schwartz-class, to obtain

$$(\phi *' f)(u)=\sum_{j\in\mathbb{Z}^d} f(j)\,\phi(\cdot-j)(u)=\sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d} f(j)\,\phi(t-j)\,u(t)\,dt.$$

With $\psi(x):=f(x)\int_{\mathbb{R}^d}\phi(t-x)\,u(t)\,dt$, $x\in\mathbb{R}^d$, we have

$$(\phi *' f)(u)=\sum_{j\in\mathbb{Z}^d}\psi(j).$$

Our next task is to apply the Poisson summation formula to the function ψ. To this end we examine properties of ψ as follows. By the Taylor formula, $f=\sum_\beta (1/\beta!)\,f^{(\beta)}(\cdot-t)\,t^\beta$. It follows that

$$\psi(x)=\sum_\beta\frac{1}{\beta!}\int_{\mathbb{R}^d} f^{(\beta)}(x-t)\,\phi(t-x)\,t^\beta u(t)\,dt. \tag{3.1}$$

Due to the decay property of ϕ and u it is clear from (3.1) that (i) $\psi\in L_1(\mathbb{R}^d)$, (ii) $\sum_{j\in\mathbb{Z}^d}\sup_{x\in[0,1)^d}|\psi(x-j)|$ is finite, and (iii) ψ is continuous. We also claim that (iv) the sum $\sum_{j\in\mathbb{Z}^d}|\hat{\psi}(2\pi j)|$ is finite. For, from (3.1),

$$\hat{\psi}(2\pi j)=\sum_\beta\frac{1}{\beta!}\,(f^{(\beta)}\phi(-\cdot))\hat{}(2\pi j)\,\hat{u}_\beta(2\pi j),\qquad j\in\mathbb{Z}^d,$$

with $u_\beta(t):=t^\beta u(t)$. The claim (iv) holds because the first factor in the above sum is bounded for all j while $u_\beta\in\mathcal{S}$ implies that the sum $\sum_{j\in\mathbb{Z}^d}|\hat{u}_\beta(2\pi j)|$ is finite. It is easy to prove that the conditions (i)–(iv) are sufficient for the Poisson summation formula

$$\sum_{j\in\mathbb{Z}^d}\psi(x+j)=\sum_{j\in\mathbb{Z}^d}\hat{\psi}(2\pi j)\,e^{-i2\pi jx}$$

to hold for every $x\in\mathbb{R}^d$, as was varified e.g., in [Ross]. Thus,

$$(\phi *' f)(u)=\sum_{j\in\mathbb{Z}^d}\psi(j)=\sum_{j\in\mathbb{Z}^d}\hat{\psi}(2\pi j).$$

With $\bar{g} := g(-\cdot)$ for a function g, this amounts to

$$(\phi *' f)(u) = \sum_{j \in \mathbb{Z}^d} f(-iD)(\hat{\bar{\phi}}\hat{u})(2\pi j). \tag{3.2}$$

Invoking the Leibniz-Hömander identity and the inverse Fourier transform formula for $u \in \mathscr{S}$, we compute

$$\begin{aligned}\widehat{\phi *' f}(u) &= (\phi *' f)(\hat{u})\\ &= (2\pi)^d \sum_{j \in \mathbb{Z}^d} \sum_{\beta} \frac{1}{\beta!} f^{(\beta)}(-iD)\, \hat{\bar{\phi}}(2\pi j)(-iD)^{\beta}\, \bar{u}(2\pi j).\end{aligned}$$

Let u_0 be an element of $\mathscr{S}$ such that $u_0(x) = 1$ if $\|x\| \leqslant 1/4$ and $u_0(x) = 0$ if $\|x\| \geqslant 1/3$. For each $\alpha \in \mathbb{Z}^d_+$ and $v \in \mathbb{Z}^d$, set

$$u_{\alpha, v} := (\cdot + 2\pi v)^{\alpha}\, u_0(\cdot + 2\pi v)/\beta!.$$

Then $(-iD)^{\alpha}\, \bar{u}_{\alpha, v}(2\pi j) = 1$ for $\alpha = \beta$ and $v = j$, and it equals zero otherwise. It follows that

$$\widehat{\phi *' f}(u_{\alpha, v}) = (2\pi)^d \frac{1}{\alpha!} f^{(\alpha)}(-iD)\, \hat{\bar{\phi}}(2\pi v),$$

for all $\alpha \in \mathbb{Z}^d_+$ and $v \in \mathbb{Z}^d$.

We are now ready to prove the lemma. If (a) is true, then the support of $\phi *' f$ is at the origin. Therefore (b) follows from (3.3). If (b) is true, then it follows from (3.2) that

$$\begin{aligned}(\phi *' f)(u) &= f(-iD)\, \widehat{\bar{\phi} *' u}(0)\\ &= \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} \phi(y - x)\, u(y)\, dy\, dx = (\phi * f)(u).\end{aligned}$$

This shows (c). The implication from (c) to (a) is trivial. ∎

Proof of Theorem 2.1. For the implication (a) ⇒ (b) we assume that $\phi *' (\mu * f) = f$ for all $f \in \Pi_K$. Here we denote by $\mu * g$ the convolution of the measure μ and a function g, i.e., $\int_{\mathbb{R}^d} g(\cdot - t)\, d\mu(t)$. By Lemma 3.1 (c) and the Fubini formula, we have

$$f = \phi *' (\mu * f) = \phi * (\mu * f) = \phi_\mu * f \qquad \text{for} \quad f \in \Pi_K, \tag{3.4}$$

with $\phi_\mu := \mu * \phi$. The function on the right-most side equals $\int_{\mathbb{R}^d} f(\cdot - y)\, \hat{\phi}_\mu(y)\, dy$. Since by the Taylor formula $f(\cdot - y) = \sum_{\beta \in \mathbb{Z}^d_+} 1/\beta!(-y)^{\beta} f^{(\beta)}$, we have

$$\phi_\mu * f = \sum_{\beta \in \mathbb{Z}_+^d} \frac{1}{\beta!} \int_{\mathbb{R}^d} (-y)^\beta \phi_\mu(y)\, dy\, f^{(\beta)}$$
$$= \sum_{\beta \in \mathbb{Z}_+^d} \frac{1}{\beta!} (-iD)^\beta \hat{\phi}_\mu(0) f^{(\beta)} = f. \tag{3.5}$$

Therefore $\hat{\phi}_\mu(0) = 1$ and $(-iD)^\beta \hat{\phi}_\mu(0) = 0$ for $\beta \in K \backslash \{0\}$, with $\hat{\phi}_\mu = \hat{\phi}\hat{\mu}$. We still need to show that for each fixed $\beta \in K$, $D^\beta \hat{\phi}(2\pi j) = 0$ for all $j \in \mathbb{Z} \backslash \{0\}$. This will follow immediately from Lemma 3.1 (c), if we can show that for the polynomial $f_\beta(x) = x^\beta$, $x \in \mathbb{R}^d$, we have $\phi *' f_\beta \in \Pi_K$. To this end, we note that for any $f \in \Pi_K$ it follows from the definition of $\mu * f$ and the Taylor formula that

$$\mu * f = \sum_{\beta \in \mathbb{Z}_+^d} \int_{\mathbb{R}^d} (-y)^\beta\, d\mu(y)\, f^{(\beta)} = \sum_{\beta \in \mathbb{Z}_+^d} (-iD)^\beta \hat{\mu}(0) f^{(\beta)}.$$

If we regard $\mu * f$ as a tempered distribution, then for any test function $u \in \mathcal{S}$

$$\widehat{\mu * f}(u) = \sum_{\beta \in \mathbb{Z}_+^d} (-iD)^\beta \hat{\mu}(0) f^{(\beta)}(-iD)\, u(0) = f(-iD)(\hat{\mu} u)(0).$$

For a fixed $\beta \in K$, if we choose

$$g_\beta(x) := \sum_{\alpha \leqslant \beta} \binom{\beta}{\alpha} (-iD)^{\beta - \alpha} \left(\frac{1}{\hat{\mu}}\right)(0)\, x^\alpha, \qquad \text{for} \quad x \in \mathbb{R}^d,$$

then clearly for every $u \in \mathcal{S}$ we have

$$\widehat{\mu * g_\beta}(u) = g_\beta(-iD)(\hat{\mu} u)(0)$$
$$= \sum_{\alpha \leqslant \beta} \binom{\beta}{\alpha} (-iD)^{\beta - \alpha} \left(\frac{1}{\hat{\mu}}\right)(0)(-iD)^\alpha (\hat{\mu} u)(0) = (-iD)^\beta u(0).$$

This amounts to $\mu * g_\beta = f_\beta$. Hence $\phi *' f_\beta = \phi *' (\mu * g_\beta) = g_\beta \in \Pi_K$. This completes the proof of the implication (a) $\Rightarrow$ (b).

For the converse statement we note that similarity to (3.4) and (3.5),

$$\phi *' (\mu * f) = \phi_\mu * f = \sum_{\beta \in \mathbb{Z}_+^d} \frac{1}{\beta!} (-iD)^\beta \hat{\phi}_\mu(0) f^{(\beta)},$$

from which the implication (b) $\Rightarrow$ (a) follows. ∎

4. Approximation Power of Quasi-Interpolants $Q_{\phi,\lambda}$

In this section we prove Theorem 2.2. We first derive a Taylor's formula with integral remainder in the coordinate-wise setting, which plays an important role in the proof of Theorem 2.2 and is of independent interest (also cf. [DDS] for Taylor polynomials and estimates of their remainders in the coordinatewise setting).

For a lower set K and $\beta \in \mathbb{Z}_+^d$, let β_K be the element of $\mathbb{Z}_+^d$ given by

$$(\beta_K)_i := \begin{cases} \beta_i, & \text{if } \beta - e^i \in K; \\ 0, & \text{otherwise.} \end{cases}$$

Here and throughout this section we always assume the index i runs over the integers $1, ..., d$ in each of its appearances.

Lemma 4.1. (The Taylor Remainder Formula). *For any lower set K and any function f, which is sufficiently smooth, we have*

$$f(x) = \sum_{\alpha \in K} \frac{1}{\alpha!} D^\alpha f(0)\, x^\alpha + \sum_{\beta \in \partial K} |\beta_K| \int_0^1 \frac{1}{\beta!} D^\beta f(sx)\, x^\beta (1-s)^{|\beta|-1}\, ds, \qquad x \in \mathbb{R}^d. \tag{4.1}$$

Proof. The proof proceeds by induction on the cardinality $|K|$ of K. When $|K| = 0$, we have $K = \{0\}$ and $\partial K = E := \{e^i\text{: all } i\}$, for which (4.1) is obviously valid. Now suppose that (4.1) holds for $|K| = m-1$, $m \geqslant 1$.

To deal with the case $|K| = m$, choose $\gamma \in K$ so that $|\gamma| = \max_{\alpha \in K} |\alpha|$, and let $K' := K \setminus \{\gamma\}$. Then K' is also a lower set, whose extension is denoted by K'^+. For convenience we denote for any $A \subset \mathbb{Z}_+^d$

$$\Lambda_A f(x) := \sum_{\alpha \in A} \frac{1}{\alpha!} D^\alpha f(0)\, x^\alpha, \qquad x \in \mathbb{R}^d,$$

and for any $\beta \in \mathbb{Z}_+^d$

$$R_\beta f(x) := \int_0^1 \frac{1}{\beta!} D^\beta f(sx)\, x^\beta (1-s)^{|\beta|-1}\, ds, \qquad x \in \mathbb{R}^d.$$

Since $|K'| = m-1$, by the induction hypotheses, we have

$$f = \Lambda_{K'} f + \sum_{\beta \in \partial K'} |\beta_{K'}|\, R_\beta f. \tag{4.2}$$

Note that $\partial K'$ agrees with ∂K, except for the point γ and possibly except for points in $E_\gamma := \gamma + E$, or precisely we have the disjoint union

$$\partial K' = (\partial K \backslash E_\gamma) \cup (\partial K' \cap E_\gamma) \cup \{\gamma\}.$$

It follows that

$$\sum_{\beta \in \partial K'} |\beta_{K'}| \, R_\beta f = \sum_{\beta \in \partial K \backslash E_\gamma} |\beta_{K'}| \, R_\beta f + \sum_{\beta \in \partial K' \cap E_\gamma} |\beta_{K'}| \, R_\beta f + |\gamma_{K'}| \, R_\gamma f.$$

It is easy to verify that for each $\beta \in \mathbb{Z}_+^d$, the vector $\beta_{K'}$ has the following properties: $\beta_{K'} = \beta_K$ if $\beta \in \partial K \backslash E_\gamma$; $\beta_{K'} = 0$ if $\beta \in E_\gamma \backslash \partial K'$; $\beta_{K'} = \gamma$ if $\beta = \gamma$. These properties yield

$$\sum_{\beta \in \partial K'} |\beta_{K'}| \, R_\beta f = \sum_{\beta \in \partial K \backslash E_\gamma} |\beta_K| \, R_\beta f + \sum_{\beta \in E_\gamma} |\beta_{K'}| \, R_\beta f + |\gamma| \, R_\gamma f. \tag{4.3}$$

Applying integration by parts to the last term above, we get

$$\begin{aligned} &|\gamma| \, R_\gamma f(x) \\ &\quad = \Lambda_{\{\gamma\}} f(x) + \sum_{i=1}^{d} \int_0^1 \frac{1}{\gamma!} D^{\gamma + e^i} f(sx) \, x^{\gamma + e^i} (1-s)^{|\gamma|} \, ds, \qquad x \in \mathbb{R}^d. \end{aligned}$$

Since $1/\gamma! = (\gamma_i + 1)/((\gamma + e^i)!)$ and $|\gamma| = |\gamma + e^i| - 1$ for all i, it follows that

$$|\gamma| \, R_\gamma f = \Lambda_{\{\gamma\}} f + \sum_{i=1}^{d} (\gamma_i + 1) \, R_{\gamma + e^i} f = \Lambda_{\{\gamma\}} f + \sum_{\beta \in E_\gamma} \rho(\beta) \, R_\beta f,$$

where $\rho(\beta) := \beta_i$ for $\beta = \gamma + e^i \in E_\gamma$. For each $\beta = \gamma + e^{i_0} \in E_\gamma$, we have $\rho(\beta) = \beta_{i_0}$ and

$$\begin{aligned} |\beta_K| &= \sum_{j \in \{i : \beta - e^i \in K\}} \beta_j = \sum_{i_0 \neq j \in \{i : \beta - e^i \in K\}} \beta_j + \beta_{i_0} \\ &= \sum_{j \in \{i : \beta - e^i \in K'\}} \beta_j + \beta_{i_0} = |\beta_{K'}| + \rho(\beta). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{\beta \in E_\gamma} |\beta_{K'}| \, R_\beta f + |\gamma| \, R_\gamma f &= \Lambda_{\{\gamma\}} f + \sum_{\beta \in E_\gamma} |\beta_{K'}| \, R_\beta f + \sum_{\beta \in E_\gamma} \rho(\beta) \, R_\beta f \\ &= \Lambda_{\{\gamma\}} f + \sum_{\beta \in E_\gamma} |\beta_K| \, R_\beta f. \end{aligned}$$

Putting this and (4.3) into (4.2), we obtain (4.1). ∎

To deal with L_p approximation, we recall a function mollification technique employed in [JL] as follows. Let χ be an element of $C^{\infty}(\mathbb{R}^d)$ such that $\operatorname{supp}\chi \subset [-1, 1]^d$, $\chi \geqslant 0$ and $\int \chi = 1.$, and let $m \in \mathbb{Z}_+$ be sufficiently large. For a given locally integrable function f, set

$$Jf(x) := \int_{\mathbb{R}^d} (f - \nabla_u^m f)(x)\, \chi(u)\, du, \qquad x \in \mathbb{R}^d,$$

where $\nabla_u^m = \nabla_u(\nabla_u^{m-1})$ and $\nabla_u f = f - f(\cdot - u)$. Clearly the operator J commutes with difference and differential operators. The following lemma is an immediate consequence of [JL, Theorems 3.1 and 3.3]. We denote by $g_|$ the restriction of a continuous function g to $\mathbb{Z}^d$, i.e., $g_| := g_{|\mathbb{Z}^d}$.

Lemma 4.2. *For $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty)$, $Jf \in C^{\infty}(\mathbb{R}^d)$. Moreover, there exists a constant C depending only on k and d such that*

(i) $\|(Jf)_|\|_{l_p(\mathbb{Z}^d)} \leqslant C \|f\|_{L_p(\mathbb{R}^d)}$, *for all* $f \in L_p(\mathbb{R}^d)$;

(ii) $\|f - Jf\|_{L_p(\mathbb{R}^d)} \leqslant C |f|_{m,p}$, *for all* $f \in W_{m,p}(\mathbb{R}^d)$.

Here, following the notation used in [A], we denote by $W_{m,p}(\mathbb{R}^d)$ the Sobolev space consisting of functions f for which $\sum_{|\beta| \leqslant m} \|D^{\beta} f\|_p < \infty$, and by $|f|_{m,p}$ the semi-norm of f given by

$$|f|_{m,p} := \sum_{|\beta| = m} \|D^{\beta} f\|_{L_p(\mathbb{R}^d)}.$$

Similarly we shall also denote $|f|_{\partial K,p} := \sum_{\beta \in \partial K} \|D^{\beta} f\|_{L_p(\mathbb{R}^d)}$.

We consider now the Taylor polynomials centered at $\zeta \in \mathbb{R}^d$,

$$\Lambda_{\zeta} f = \Lambda_{\zeta,K} f = \sum_{\alpha \in K} \frac{D^{\alpha} f(\zeta)}{\alpha!} (\cdot - \zeta)^{\alpha},$$

for a sufficiently smooth function f. Let $R_{\zeta,K}$ be the operator giving remainders corresponding to $\Lambda_{\zeta,K}$.

Lemma 4.3. *Given lower set K and $1 \leqslant p < \infty$, for any sufficiently smooth f with $|f|_{\partial K,p}$ finite and for any $x, y \in \mathbb{R}^d$, it follows that*

$$\left(\sum_{j \in \mathbb{Z}^d} |R_{x+j,K} Jf(y+j)|^p \right)^{1/p} \leqslant C |f|_{\partial K,p}\, w(y - x),$$

where

$$w(x) := \max_{\beta \in \partial K} (1+|x_1|)^{\beta_1} \cdots (1+|x_d|)^{\beta_d}, \qquad \text{for} \quad x = (x_1, ..., x_d) \in \mathbb{R}^d,$$

and C is a constant independent of x, y, p and f.

Proof. It follows from the previous lemma that

$$\begin{aligned} R_{x+j,K} Jf(y+j) \\ = \int_0^1 \sum_{\beta \in \partial K} \frac{|\beta_K|}{\beta!} (1-s)^{|\beta|-1} (y-x)^\beta D^\beta Jf(x+j+s(y-x))\, ds. \end{aligned}$$

If we keep $x, y \in \mathbb{R}^d$ fixed and use the notation

$$L_s := \sum_{\beta \in \partial K} \frac{|\beta_K|}{\beta!} (1-s)^{|\beta|-1} (y-x)^\beta \tau^{x+s(y-x)} D^\beta,$$

with τ^z being the translation operator: $g \mapsto g(\cdot + z)$, then the above remainder formula may be abbreviated

$$c(j) := R_{x+j,K} Jf(y+j) = \int_0^1 (L_s Jf)(j)\, ds, \qquad j \in \mathbb{Z}^d.$$

For each $0 \leq s \leq 1$, the operator L_s is made up of translation and differentiation operators, hence commutes with J. Therefore, applying Minkowski's inequality and Lemma 4.2(i) we have

$$\begin{aligned} \|c\|_{l_p(\mathbb{Z}^d)} &\leq \int_0^1 \|(L_s Jf)_|\|_{l_p(\mathbb{Z}^d)}\, ds = \int_0^1 \|(JL_s f)_|\|_{l_p(\mathbb{Z}^d)}\, ds \\ &\leq C \int \|L_s f\|_{L_p(\mathbb{R}^d)}\, ds. \end{aligned}$$

Because $|(y-x)^\beta| \leq w(y-x)$ for $\beta \in \partial K$, it follows from the definition of L_s that $\|L_s f\|_{L_p(\mathbb{R}^d)} \leq C' w(x-y) |f|_{\partial K, p}$, where C' is a constant independent of x, y, p and f. The proof is complete. ∎

Proof of Theorem 2.2. We sketch the proof following the line of [L₁] with some necessary modification and supplement. We first deal with the special case $h = (1, ..., 1) \in \mathbb{R}^d$, and then with the general case by scaling.

Let us begin by showing the operator $Q := Q_{\phi, \lambda}: f \to \sum_{j \in \mathbb{Z}^d} \lambda f(\cdot + j)\, \phi(\cdot - j)$ is bounded, denoting the bound by $\|Q\|_p$. For any $f \in L_p(\mathbb{R}^d)$, $Qf = \phi *' (\lambda * f)$. By [JM, Theorem 2.1] and the assumption (2.2),

$$\|Qf\|_{L_p(\mathbb{R}^d)} \leq |\phi|_p \, \|(\lambda * f)_|\|_{l_p(\mathbb{Z}^d)} \leq C_p \, |\phi|_p \, \|f\|_{L_p(\mathbb{R}^d)}.$$

For a given $f \in W_{m,p}$, it follows from Lemma 4.2 (ii) that

$$\|f - Qf\|_{L_p(\mathbb{R}^d)} \leqslant (1 + \|Q\|_p)\|f - Jf\|_{L_p(\mathbb{R}^d)} + \|Jf - QJf\|_{L_p(\mathbb{R}^d)}$$
$$\leqslant C(1 + \|Q\|_p)|f|_{m,p} + \|Jf - QJf\|_{L_p(\mathbb{R}^d)}.$$

Here and herafter C and $C_1, C_2, \ldots$ are constants depending only on K, ϕ and d.

To estimate the term $\|Jf - QJf\|_{L_p(\mathbb{R}^d)}$ we associate a sequence a_x to each $x \in I := [0, 1)^d$ by setting

$$a_x(\alpha) := Jf(x + \alpha) - QJf(x + \alpha), \qquad \alpha \in \mathbb{Z}^d,$$

because in this way we have

$$\|Jf - QJf\|_{L_p(\mathbb{R}^d)} = \left(\int_I \|a_x\|^p_{l_p(\mathbb{Z}^d)}\right)^{1/p}$$

with the usual change on the right when $p = \infty$. By the polynomial reproducing property of Q, we compute that for each $x \in I$ and $\alpha \in \mathbb{Z}^d$,

$$\begin{aligned} a_x(\alpha) &= \Lambda_{x+\alpha} Jf(x + \alpha) - QJf(x + \alpha) = Q(\Lambda_{x+\alpha} Jf - Jf)(x + \alpha) \\ &= \sum_{j \in \mathbb{Z}^d} \lambda R_{x+\alpha} Jf(\cdot + j)\, \phi(x + \alpha - j) \\ &= \sum_{v \in \mathbb{Z}^d} \lambda R_{x+\alpha} Jf(\cdot + v + \alpha)\, \phi(x - v) \\ &= \sum_{v \in \mathbb{Z}^d} \int_{\mathbb{R}^d} R_{x+\alpha} Jf(u + v + \alpha)\, g_\lambda(u)\, du\, \phi(x - v). \end{aligned}$$

In the following we apply Minkowski's inequality, Lemma 4.3 and the fact $w(u + v - x) \leqslant w(u)\, w(x - v)$ to estimate $\|a_x\|_{l_p(\mathbb{Z}^d)}$ for each $x \in I$:

$$\begin{aligned} &\|a_x\|_{l_p(\mathbb{Z}^d)} \\ &\quad\leqslant \sum_{v \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \left(\sum_{\alpha \in \mathbb{Z}^d} |R_{x+\alpha} Jf(u + v + \alpha)|^p\right)^{1/p} |g_\lambda(u)|\, du\, |\phi(x - v)| \\ &\quad\leqslant \sum_{v \in \mathbb{Z}^d} \int_{\mathbb{R}^d} C_1 |f|_{\partial K, p}\, w(u + v - x) |g_\lambda(u)|\, du\, |\phi(x - v)| \\ &\quad\leqslant C_1 |f|_{\partial K, p} \int_{\mathbb{R}^d} w(u) |g_\lambda(u)|\, du \sum_{v \in \mathbb{Z}^d} w(x - v) |\phi(x - v)|. \end{aligned}$$

$$\|Jf - QJf\|_{L_p(\mathbb{R}^d)} \leqslant C_2 |f|_{\partial K, p}\, |w\phi|_p.$$

To summarize we have proven

$$\|f - Qf\|_{L_p(\mathbb{R}^d)} \leqslant C(1 + \|Q\|_p)|f|_{m,p} + C_2|f|_{\partial K, p}\, |w\phi|_p.$$

Theorem 2.2 follows from the above inequality by scaling. ∎

Before closing this section we make the following remarks, which show how one can construct a quasi-interpolant working for L_p spaces from a regular Borel measure on $C_0(\mathbb{R}^d)$, say, a linear combination of point evaluation functionals that fail to apply to L_p functions in general. Let K be a lower set.

(1) If ϕ satisfies the Strang–Fix conditions on lower set K, then there is a finitely supported sequence $\{a_v\}_{\mu \in \mathbb{Z}^d}$ such that $\phi *' (\mu * f) = f$ for $f \in \Pi_K$, where the measure μ is given by the rule $\int_{\mathbb{R}^d} g\, d\mu = \sum_{v \in \mathbb{Z}^d} a_v g(v)$ for $g \in C_0$. This can be proven by a similar argument used in the proof for [Proposition 4.1, BrCW].

(2) For any regular Borel measure μ satisfying the decay condition that $\Pi_K \subset L_1(|\mu|)$, the linear functional λ given by

$$\lambda g := \int_{\mathbb{R}^d} Jg\, d\mu, \qquad g \in L_p(\mathbb{R}^d),$$

$(1 \leqslant p < \infty)$ is well defined on $L_p(\mathbb{R}^d)$ (by Lemma 4.2) and satisfies the same decay condition $\Pi_K \subset L_1(|\mu_\lambda|)$, because J is a compactly supported operator. Furthermore, if $\phi *' (\mu * f) = f$ for all $f \in \Pi_K$, then $\phi *' (\lambda * f) = f$ for all $f \in \Pi_K$, since $Jf = f$ for all $f \in \Pi_K$.

5. Proof of Theorem 2.3

In this section we prove our inverse result Theorem 2.3, namely that the degree of the polynomial reproducing property of a quasi-interpolant Q is at least the approximation order provided by Q.

The following lemma is needed for the proof of Theorem 2.3. A remark is in order: The converse of this lemma, hence the necessity of the concavity in Theorem 2.3, is also valid ([BrCW]).

Lemma 5.1. *Let K be a concave lower set. For any $\alpha \in K$, the quotient*

$$q_h := h^\alpha / \max\{h^\gamma : \gamma \in \partial K\}$$

is unbounded as $\|h\| \to 0$.

Proof. The proof proceeds along the lines of [BrCW]. Let K be concave and $\alpha \in K$. We can find a hyperplane in $\mathbb{R}^d$ that separates α from the convex hull $\operatorname{conv}(\mathbb{Z}_+^d \backslash K)$. If ε is a normal vector of the separating plane with $\varepsilon \cdot (x-\alpha) > 0$ for $x \in \operatorname{conv}(\mathbb{Z}_+^d \backslash K)$, then ε has positive components ε_i, because K is finite and hence any vector $x = \theta e^i \in \mathbb{Z}_+^d \backslash K$ for sufficiently large $\theta > 0$. The unboundedness of $q_h = \min h^{\alpha - \gamma}$ for $\gamma \in \partial K$ now follows by letting $h = (h_1, ..., h_d)$ with $h_i = e^{-t\varepsilon_i}$, where $t > 0$ and $i = 1, ..., d$. Then $\|h\| \to 0$ as $t \to \infty$, and we have

$$q_h = \min\{e^{t\varepsilon \cdot (\gamma - \alpha)}: \gamma \in \partial K\} \to \infty,$$

since $\partial K \subset \operatorname{conv}(\mathbb{Z}_+^d \backslash K)$. ∎

Proof of Theorem 2.3. We deal only with the case $1 \leqslant p < \infty$, as the case $p = \infty$ follows in a similar (easier) way. It suffices to show that $Qf_\beta = f_\beta$ for the polynomials $f_\beta(x) = x^\beta$, $\beta \in K$. The proof proceeds by induction on $\beta \in K$.

For each $r > 0$ let R_r be the cube $[0, r]^d$. Clearly $R_{r/2}$ is a subset of the ball $B_r(0)$. When $\beta = 0$, $\lambda f_\beta(\cdot + j) = \lambda(1) = \hat{\lambda}(0)$ for all $j \in \mathbb{Z}^d$. Under the change of variables $y = (x - \xi)/h$ (assuming that $h = (h_1, ..., h_d)$ has positive coordinates),

$$\begin{aligned} J_{h,0} &:= \int_{B_r(\xi)} \Big|1 - \sum_{j \in \mathbb{Z}^d} \lambda(1)\, \phi(x/h - j)\Big|^p dx \\ &\geqslant \int_{\xi + R_{r/2}} \Big|1 - \sum_{j \in \mathbb{Z}^d} \lambda(1)\, \phi(x/h - j)\Big|^p dx \\ &= h_1 \cdots h_d \int_{[0,(r/2h_1)] \times \cdots \times [0,(r/2h_d)]} \Big|1 - \sum_{j \in \mathbb{Z}^d} \hat{\lambda}(0)\, \phi(y + \xi/h - j)\Big|^p dy. \end{aligned}$$

With $\lceil t \rceil$ being the greatest integer part of a real number t, it follows from the periodicity of the integrand in the above that

$$\begin{aligned} J_{h,0} &\geqslant h_1 \cdots h_d \left\lceil \frac{r}{2h_1} \right\rceil \cdots \left\lceil \frac{r}{2h_1} \right\rceil \int_{R_1} \Big|1 - \sum_{j \in \mathbb{Z}^d} \hat{\lambda}(0)\, \phi(y - j)\Big|^p dy \\ &\geqslant (r/4) \int_{R_1} \Big|1 - \sum_{j \in \mathbb{Z}^d} \hat{\lambda}(0)\, \phi(y - j)\Big|^p dy, \end{aligned}$$

where for the last inequality we have used the fact that $\lceil r/2h_i \rceil \geqslant (r/4h_i)$ when $h_i \leqslant r/4$. Note that the right-most quantity in the above display does not depend on h. However, from the approximation property assumed for the quasi-interpolant Q it follows that $J_{h,0} \to 0$ as $\|h\| \to 0$. Hence $1 = f_0 = Qf_0$.

Now we assume that $0 \neq \beta \in K$ and that $Qf_\alpha = f_\alpha$ for all $\alpha < \beta$. Similarly to the case of $J_{h,0}$, we have

$$\begin{aligned} J_{h,\beta} &:= \int_{B_r(\xi)} \Big|x^\beta - \sum_{j \in \mathbb{Z}^d} \lambda(h(\cdot + j))^\beta \phi(xh - j)\Big|^p \, dx \\ &\geq (r/4)^d \int_{R_1} \Big|(hy + \xi)^\beta - \sum_{j \in \mathbb{Z}^d} \lambda(h(\cdot + j))^\beta \phi(y + \xi/h - j)\Big|^p \, dx \\ &= (r/4)^d h^{p\beta} \int_{R_1} \Big|(y + \xi/h)^\beta - \sum_{j \in \mathbb{Z}^d} \lambda(\cdot + j)^\beta \phi(y + \xi/h - j)\Big|^p \, dx, \end{aligned}$$

where we have used the linearity of λ in the last inequality. To see the periodicity of the integrand in the right-most integral, we expand the following periodic function of $z \in \mathbb{R}^d$ using the binomial identity:

$$\begin{aligned} \rho(z) &:= \sum_{j \in \mathbb{Z}^d} \lambda(\cdot + j - z)^\beta \phi(z - j) \\ &= \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} (-z)^{\beta - \alpha} \sum_{j \in \mathbb{Z}^d} \lambda(\cdot + j)^\alpha \phi(z - j). \end{aligned}$$

From the induction hypothesis,

$$\begin{aligned} \rho(z) &= \sum_{\alpha < \beta} \binom{\beta}{\alpha} (-z)^{\beta - \alpha} (z)^\alpha + \sum_{j \in \mathbb{Z}^d} \lambda(\cdot + j)^\beta \phi(z - j) \\ &= -z^\beta + \sum_{j \in \mathbb{Z}^d} \lambda(\cdot + j)^\beta \phi(z - j). \end{aligned} \tag{5.1}$$

Putting all of these together, we obtain

$$J_{h,\beta} \geq (r/4)^d h^{p\beta} \int_{R_1} |\rho(y + \xi/h)| \, dy = (r/4)^d h^{p\beta} \int_{R_1} |\rho(y)| \, dy.$$

However, from the approximation property of Q it follows that

$$(r/4)^{d/p} h^\beta \left(\int_{R_1} |\rho(y)|^p \, dy \right)^{1/p} \leq (J_{h,\beta})^{1/p} = \mathcal{O}(\max\{h^\gamma : \gamma \in \partial K\}.$$

By virtue of Lemma 5.1 and the concavity of K, we have $\rho(y) = 0$ for all $y \in \mathbb{R}^d$. Since by (5.1) $\rho = f_\beta - Qf_\beta$, we see that $Qf_\beta = f_\beta$. This completes the induction.

REFERENCES

[A] R. A. ADAMS, "Sobolev Spaces," Academic Press, Boston, 1975.

[B1] C. DE BOOR, Quasiinterpolants and approximation power of multivariate splines, *in* "Computation of Curves and Surfaces" (W. Dahmen, M. Gasca, and C. A. Micchelli, Eds.), pp. 313–345, Kluwer, Dordrecht, 1990.

[B2] C. DE BOOR, Approximation order without quasi-interpolants, *in* "Approximation Theory VII" (E. W. Cheney, C. K. Chui, and L. L. Schumaker, Eds.), pp. 1–18, Academic Press, New York, 1993.

[BDR] C. DE BOOR, R. DEVORE, AND A. RON, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, CMS TSR #92-02, University of Wisconsin–Madison, 1992.

[BR] C. DE BOOR AND A. RON, Fourier analysis of the approximation power of principal shift-invariant spaces, *Constr. Approx.* **8** (1992), 427–462.

[Buh] M. D. BUHMAN, Multivariate cardinal interpolation with radial basis functions, *Constr. Approx.* **6** (1990), 225–255.

[BrCW] H. G. BURCHARD, C. K. CHUI, AND J. D. WARD, Polynomial degree and approximation order, CAT Rep. 175, Texas A&M University, 1988.

[Ch] C. K. CHUI, "Multivariate Splines," CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 54, SIAM, Philadelphia, 1988.

[CJW] C. K. CHUI, K. JETTER, AND J. D. WARD, Cardinal interpollation by multivariate splines, *Math. Comp.* **48** (1987), 711–724.

[CL] C. K CHUI AND M. J. LAI, A multivariate analog of Marsden's identity and a quasi-interpolation scheme, *Constr. Approx.* **3** (1987), 111–122.

[DDS] W. DAHMEN, R. DEVORE, AND K. SCHERER, Multidimensional spline approximation, *SIAM J. Numer. Anal.* **17** (1980), 380–402.

[HL] E. J. HALTON AND W. A. LIGHT, On local and controlled approximation order, *J. Approx. Theory* **72** (1993), 268–277.

[JL] R. Q. JIA AND J. J. LEI, Approximation by multiinteger translates of functions having global support, *J. Approx. Theory* **72** (1993), 2–23.

[JM] R. Q. JIA AND C. A. MICCHELLI, Using the refinement equations for the construction of pre-wavelets, II. Powers of two, *in* "Curves and Sufaces" (P. J. Laurent, A. le Méhauté, and L. L. Schumaker, Eds.), pp. 209–246, Academic Press, Boston, 1991.

[L1] J. J. LEI, On approximation by translates of globally supported functions, *J. Approx. Theory* **77** (1994), 123–138.

[L2] J. J. LEI, L_p-approximation by certain projection operators, *J. Math. Anal. Appl.* **185** (1994), 1–14.

[LC] W. A. LIGHT AND E. W. CHENEY, Quasi-interpolation with translates of a function having non-compact support, *Constr. Approx.* **8** (1992), 35–48.

[Light] W. A. LIGHT, Using radial functions on compact domains, *in* "Wavelets, Images, and Surface Fitting" (P. J. Laurent, A. le Méhauté, and L. L. Schumaker, Eds.), pp. 351–370, AK Peters, Wellesley, MA, 1994.

[Ross] K. ROSS, Poisson summation, *manuscript*, 1991.